Model of the Newtonian cosmology: Symmetries, invariant and partialy invariant solutions

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Abstract

Symmetry group of the equation system of ideal nonrelativistic self-gravitating fluid with zero pressure is calculated. Submodel invariant under the subgroup of rotations SO(3) is built and symmetry group of the factorsystem is calculated. A particular analytical invariant solution of the factorsystem is obtained.

Keywords: Newtonian cosmology, ideal self-gravitating fluid, Lie point symmetries, invariant solutions

1. Introduction

Model of the Newtonian cosmology is basic in the study of large-scale structure of the Universe [1, 2]. The model is a system of equations of ideal nonrelativistic self-gravitating fluid with zero pressure, density ρ, velocity $\overrightarrow{v}$ and gravitational potential Φ [2]

$$\begin{cases} \frac{\partial \rho}{\partial t} + div(\rho \overrightarrow{v}) = 0, \\ \frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v}\nabla)\overrightarrow{v} + grad(\Phi) = 0, \\ \Delta\Phi = 4\pi\gamma\rho. \end{cases} \tag{1}$$

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The first equation is the equation of continuity, the second — Euler equation, the third — Poisson equation, Δ — Laplace operator, γ is the gravitational constant. At the present time various approximations of the solution of system (1) are well studied, the best known of them is Zeldovich "pancakes" model [1, 3]. The aim of our work is systematically study the system of equations (1) by methods of group analysis that will yield new exact analytical solutions that not only going beyond perturbation theory but also useful for testing numerical methods.

Due to the symmetry in the model of Newtonian cosmology one can apply the SUBMODELS program, similar to L. V. Ovsyannikov's program in gas dynamics. So, the system (1) will present as a "big model" [6].

2. Lie point symmetries

Rewrite the system (1) in Cartesian coordinates in dimensionless variables:

$$\begin{cases} \rho_t + \rho(u_x + v_y + \omega_z) + u\rho_x + v\rho_y + \omega\rho_z = 0, \\ u_t + uu_x + vu_y + \omega u_z + \Phi_x = 0, \\ u_t + uv_x + vv_y + \omega v_z + \Phi_y = 0, \\ u_t + u\omega_x + v\omega_y + \omega\omega_z + \Phi_z = 0, \\ \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = \rho, \end{cases} \tag{2}$$

where x, y, z — Cartesian coordinates, t — time, u, v, ω — the velocity components.

Generator of the group will be sought in the form

$$\hat{X} = \xi^{(x)}\partial_x + \xi^{(y)}\partial_y + \xi^{(z)}\partial_z + \xi^{(t)}\partial_t + \eta^{(p)}\partial_p + \eta^{(\Phi)}\partial_\Phi + \eta^{(u)}\partial_u + \eta^{(v)}\partial_v + \eta^{(\omega)}\partial_\omega,$$

where ξ and η — the tangent vector field components, ∂ — operator of differentiation in the corresponding variable. Calculation according to standard algorithm Lie-Ovsyannikov [4] with help of package GeM [5] shows that the system allows for an infinite-dimensional algebra with the components of the tangent vector field

$$\begin{aligned}
\xi^{(x)} &= F_1(t) + (C_1 - C_4)x + C_2 y + C_{10},\\
\xi^{(y)} &= F_2(t) + (C_1 - C_4)y - C_2 x + C_6 z + C_7,\\
\xi^{(z)} &= F_3(t) + (C_1 - C_4) + C_8 t - C_3 x - C_6 y + C_9,\\
\xi^{(t)} &= -C_4 t + C_5,\\
\eta^{(p)} &= 2C_4\rho,\\
\eta^{(\Phi)} &= 2C_1\Phi - F_{1tt}x - F_{2tt}y + F_4(t) + F_{3tt}z,\\
\eta^{(u)} &= C_1 u + C_2 v + C_3\omega + F_{1t},\\
\eta^{(v)} &= C_1 v - C_2 u + C_6\omega + F_{2t},\\
\eta^{(\omega)} &= C_1\omega - C_3 u - C_6 v + F_{3t} + C_8,
\end{aligned}$$

where $F_i(t)$ — arbitrary functions, C_k — arbitrary constants. This infinite-dimensional algebra contains a 13-dimensional subalgebra, say, L_{13} with generator translations

$$\hat{X}_1 = \partial_x, \quad \hat{X}_2 = \partial_y, \quad \hat{X}_3 = \partial_z, \quad \hat{X}_4 = \partial_t, \quad \hat{X}_5 = \partial_\Phi;$$

Galilean transformations

$$\hat{X}_6 = t\partial_x + \partial_u, \quad \hat{X}_7 = t\partial_y + \partial_v, \quad \hat{X}_8 = t\partial_z + \partial_\omega;$$

rotations

$$\begin{aligned}
\hat{X}_9 = y\partial_z - z\partial_y + v\partial_\omega - \omega\partial_v, \quad \hat{X}_{10} &= z\partial_x - x\partial_z + \omega\partial_u - u\partial_\omega,\\
\hat{X}_{11} &= x\partial_y - y\partial_x + u\partial_v - v\partial_u;
\end{aligned}$$

and dilatations

$$\begin{aligned}
\hat{X}_{12} &= 2\Phi\partial_\Phi + u\partial_u + v\partial_v + \omega\partial_\omega + x\partial_x + y\partial_y + z\partial_z,\\
\hat{X}_{13} &= -2\rho\partial_\rho + x\partial_x + y\partial_y + z\partial_z + t\partial_t.
\end{aligned}$$

The corresponding commutator table for a Lie Algebra L_{13} has the form Table 1.

To realize the SUBMODEL program [6] it is necessary to calculate the optimal system of subalgebras for the 13-dimensional algebra $\hat{X}_1, ...\hat{X}_{13}$, which is the subject of a separate study. In this paper, we focus on the submodels, which is invariant under the rotation group SO(3) with generators $(\hat{X}_9, \hat{X}_{10}, \hat{X}_{11})$.

Table 1: Commutator table for L_{13}

	$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$	$\hat{X}_5$	$\hat{X}_6$	$\hat{X}_7$	$\hat{X}_8$	$\hat{X}_9$	$\hat{X}_{10}$	$\hat{X}_{11}$	$\hat{X}_{12}$	$\hat{X}_{13}$
$\hat{X}_1$	0	0	0	0	0	0	0	$-\hat{X}_3$	$\hat{X}_2$	0	0	$-\hat{X}_1$	$\hat{X}_1$
$\hat{X}_2$	0	0	0	0	0	0	$\hat{X}_3$	0	$-\hat{X}_1$	0	0	$-\hat{X}_2$	$\hat{X}_2$
$\hat{X}_3$	0	0	0	0	0	0	$\hat{X}_2$	$\hat{X}_1$	0	0	0	$-\hat{X}_3$	$\hat{X}_3$
$\hat{X}_4$	0	0	0	0	0	0	0	$-\hat{X}_6$	$\hat{X}_5$	$-\hat{X}_1$	0	$-\hat{X}_4$	0
$\hat{X}_5$	0	0	0	0	0	0	$-\hat{X}_6$	0	$-\hat{X}_4$	$-\hat{X}_2$	0	$-\hat{X}_5$	0
$\hat{X}_6$	0	0	0	0	0	0	$-\hat{X}_5$	$\hat{X}_4$	0	$-\hat{X}_3$	0	$-\hat{X}_6$	0
$\hat{X}_7$	0	$-\hat{X}_3$	$\hat{X}_2$	0	$-\hat{X}_6$	$\hat{X}_5$	0	$-\hat{X}_9$	$\hat{X}_8$	0	0	0	0
$\hat{X}_8$	$\hat{X}_3$	0	$-\hat{X}_1$	$\hat{X}_6$	0	$-\hat{X}_4$	$\hat{X}_9$	0	$-\hat{X}_7$	0	0	0	0
$\hat{X}_9$	$-\hat{X}_2$	$\hat{X}_1$	0	$-\hat{X}_5$	$\hat{X}_4$	0	$-\hat{X}_8$	$\hat{X}_7$	0	0	0	0	0
$\hat{X}_{10}$	0	0	0	$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	0	0	0	0	0	0	$\hat{X}_{10}$
$\hat{X}_{11}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\hat{X}_{12}$	$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$	$\hat{X}_5$	$\hat{X}_6$	0	0	0	0	0	0	0
$\hat{X}_{13}$	$-\hat{X}_1$	$-\hat{X}_2$	$-\hat{X}_3$	0	0	0	0	0	0	$-\hat{X}_{10}$	0	0	0

3. Invariant submodel SO(3)

Invariants of the group SO(3) are

$$\Phi, \rho, t, r = \sqrt{x^2+y^2+z^2}, \quad |\overrightarrow{v}| = \sqrt{u^2+v^2+\omega^2} \equiv U, \quad s = \overrightarrow{r}\overrightarrow{v}.$$

In view of the fact that $\overrightarrow{v} = U\frac{\overrightarrow{r}}{r}$ [7], invariant solution should be sought in the form

$$\overrightarrow{v} = \frac{\overrightarrow{r}}{r}U(t,r), \quad \rho = \rho(t,r), \quad \Phi = \Phi(t,r). \tag{3}$$

Substituting (3) in (2) one can obtain a factorsystem

$$\begin{cases} U_t + UU_r + \Phi_r = 0, \\ \rho_t + U\rho_r + \rho(\frac{2U}{r} + U_r) = 0, \\ \frac{2\Phi_r}{r} + \Phi_{rr} = \rho. \end{cases} \tag{4}$$

Factorsystem (4) allows a 3-dimensional algebra, say, L_3:

$$\begin{cases} \hat{X}_1 = 2\Phi\partial_\Phi + U\partial_U + r\partial_r, \\ \hat{X}_2 = -2\rho\partial_\rho + r\partial_r + t\partial_t, \\ \hat{X}_3 = \partial_t, \end{cases} \tag{5}$$

Table 2: Commutator table for L_3

	$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$
$\hat{X}_1$	0	0	0
$\hat{X}_2$	0	0	$-\hat{X}_3$
$\hat{X}_3$	0	$\hat{X}_3$	0

and infinite-dimensional algebra with generator $\hat{X}_\infty = F_1(t)\partial_\Phi$, where $F_1(t)$ an arbitrary function of time t. Let us find the optimal system of subalgebras of a 3-dimensional algebra (5). From the commutator table for L_3 (Table 2) one can obtain the inner automorphisms of L_3.

Consider a one-dimensional subalgebra generated by the vector $aX_1 + bX_2 + cX_3$. In case $b \neq 0$ one can nullify coordinate of c. Dividing then on b, one obtains the vector $aX_1 + X_2$, where the parameter of a is non-negative (used reflection on the first coordinate). When $b = 0$, using stretching along the third coordinate and basic transformation, one obtains the three cases: $X_1,\ X_3,\ X_1 + X_3$. So, all one-dimensional algebras in L_3 are equivalent to subalgebras with the basis vectors $aX_1 + X_2$, X_1, X_3, $X_1 + X_3$.

In any two-dimensional subalgebra, one of the basis vectors can be reduced to aforementioned vectors. Therefore, to list all the two-dimensional subalgebras it is sufficient to consider the subalgebra with one basis vector of the vectors mentioned above, the second basis vector is taken in an arbitrary manner and simplified by inner automorphisms and basic transformations.

With vector X_3 one can obtain two-dimensional subalgebras $\langle X_1, X_3\rangle$, $\langle aX_1 + X_2, X_3\rangle$, $a \geq 0$. Vector X_1 gives subalgebra $\langle X_1, X_2\rangle$. The remaining vectors give no new subalgebras. Any two-dimensional subalgebra can be reduced by inner automorphisms to subalgebras $\langle X_1, X_2\rangle$, $\langle X_1, X_3\rangle$, $\langle aX_1 + X_2, X_3\rangle$, $a \geq 0$.

As an example, consider a solution that is invariant with respect to the generator $\hat{X}_1$. Invariants are $\frac{U}{r}, \frac{\Phi}{r^2}, \rho, t$. The solution is sought in the form $U = rf_1(t), \Phi = r^2 f_2(t), \rho = \rho(t)$. The factorsystem is a system of ordinary differential equations

$$\begin{cases} f_1' + f_1^2 + 2f_2 = 0, \\ \rho' + 3f_1\rho = 0, \\ \rho = 6f_2. \end{cases} \tag{6}$$

Here $f_i' \equiv \frac{df_i}{dt}\,(i = 1,2)$, $\rho' \equiv \frac{d\rho}{dt}$. Eliminating ρ and f_2, one obtains an ordinary differential equation for the function f_1

$$f_1^{''} + 5f_1f_1^{'} + 3f_1^3 = 0, \tag{7}$$

which admits, in turn, a two-dimensional Lie algebra with generators ∂_t and $t\partial_t - f_1\partial_{f_1}$. Following the Lie algorithm [8], we find the canonical variables $u = t - \frac{1}{f_1}, \tau = \frac{1}{f_1}$. In the canonical variables, equation (7) can be solved in closed analytic form

$$\begin{cases} \frac{du}{d\tau} = F(\tau), \\ \int \frac{dF}{3F^3+14F^2+21F-10} = \ln|\tau| + C. \end{cases} \tag{8}$$

Thus, the considered submodel is exactly solvable. The obtained solution is the basis for Newtonian cosmology [3], but the group-theoretical nature of this solution was not discussed up till now in the physical literature. The construction of other invariant solutions of spherically symmetric submodel and the analysis of their physical meaning is of undoubted interest for astrophysics and is the subject of the following publication.

4. Conclusion

This article is the first in a series of papers devoted to the study of invariant and partially invariant solutions of the model of Newtonian cosmology. In subsequent papers we intend to consider the following problems: building all invariant solutions of spherically symmetric submodel SO(3); building partially invariant solution of such "Ovsyannikov's vortex" [9]; calculation optimal system subalgebras of 13-dimensional Lie algebra, admitted by the the "big model" [6].

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